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G-Majorization, Group-Induced Cone Orderings, and Reflection Groups

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ABSTRACT

A vector y is G-majorized by a vector x if y is an element of the convex hull of the orbit of x under the action of a group G. It is known that if G is a finite reflection group, then G-majorization is equivalent to a group-induced cone ordering. In this paper it is established that if for a finite subgroup G of the orthogonal group G-majorization is equivalent to a group-induced cone ordering, then G must be a reflection group.

1. INTRODUCTION

Multivariate inequalities are of interest in pure and applied mathematics. For surveys and recent developments in the area of multivariate inequalities in statistics and probability we refer e.g. to Tong (1984), Steerneman (1987), and Eaton (1987a, b). An important issue is the study of vector orderings. In this paper we will focus on a preordering known as G-majorization. It has a long history. Three important landmarks are Hardy, Littlewood, and Pólya (1973, 1st ed., 1934), Marshall and Olkin (1979), and Eaton and Perlman (1977). We shall not go into details regarding history, but mention some literature relevant to the development of the general concept of G-majorization: Mudholkar (1966), Eaton and Perlman (1977), Eaton (1982, 1984, 1987a, b), Jensen (1984), and Giovagnoli and Wynn (1985). In the literature there are various illustrative examples available, and these will not be

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reproduced in this paper. Excellent references are e.g. Eaton (1984, 1987a, b).

We have the following aim. In case G is a finite reflection group, G-majorization is equivalent to a so-called group-induced cone ordering. We derive a converse result. If for a finite group G, G-majorization is equivalent to a group-induced cone ordering, then G is a reflection group; see Section 4. In the following two sections a number of results are collected that are needed in Section 4. Section 2 is devoted to G-majorization and its relation to a group-induced cone ordering. Section 3 surveys some facts about reflection groups and fundamental regions. New results are presented in Section 4.

2. G-MAJORIZATION

Let $(V,(\cdot,\cdot))$ be a finite dimensional real inner product space, and let G be a group of bijective transformations of V. In our notation we shall not always express the dependence on G. The group G induces an equivalence relation $\approx$ on V, defined by $x \approx y$ if and only if $y = gx$ for some $g \in G$. The equivalence classes are the orbits of G:

$$O(x) = \{gx | g \in G\}, \qquad x \in V.$$

The convex hull of the orbit $O(x)$ will be denoted by $C(x)$.

The group G also induces another relation on V, which is called G-majorization. The general concept is due to Mudholkar (1966) and Eaton and Perlman (1977). It is denoted by $\prec$ and defined by

$$y \prec x \quad \Leftrightarrow \quad y \in C(x).$$

In general $\prec$ is a relation. If G is a group of linear bijections, then $\prec$ is a preordering. In the sequel G will be a closed subgroup of $\mathcal{O}(V)$, the orthogonal group; hence G is compact. This implies that both $O(x)$ and $C(x)$ are compact sets in V. It can easily be shown that the orbit $O(x)$ is equal to the set of extremal points of $C(x)$ [see Lemma 1 of Giovagnoli and Wynn (1985)]. Some other obvious results are collected in the following proposition.

PROPOSITION 2.1. *Let $x,\ z \in V$, then*

(i) $gC(x) = C(gx) = C(x)$ *for all* $g \in G$,
(ii) $z \prec x \ \Leftrightarrow\ C(z) \subset C(x)$,
(iii) $z \prec x,\ x \prec z \ \Leftrightarrow\ x \approx z$.

An element $m \in V$ is G-minimal in V if $x \prec m$ implies $x \approx m$. The G-minimal elements turn out to be precisely the G-invariant elements; see Remark 2.3 of Eaton and Perlman (1977). Define

$$M_G = \{x \in V | gx = x \quad \text{for all } \ g \in G\}; \tag{2.1}$$

then the element $m \in V$ is minimal if and only if $m \in M_G$.

In principle it could be a hard task to check whether $y \prec x$. The application of the so-called support function has proved to be very successful. The idea has been developed in Eaton (1984) and in Giovagnoli and Wynn (1985). For u, $x \in V$ define $m: V \times V \to \mathbb{R}$ by

$$m[u, x] = \sup_{g \in G} (u, gx). \tag{2.2}$$

The following proposition collects some basic results [see Equation (2.5) from Eaton (1984)].

PROPOSITION 2.2. *For* $u, x \in V$,

(i) $m[c_1 u, c_2 x] = c_1 c_2 m[u, x]$ *for* $c_1, c_2 \geqslant 0$,
(ii) $m[g_1 u, g_2 x] = m[u, x]$ *for* $g_1, g_2 \in G$,
(iii) $m[u, x] = m[x, u]$,
(iv) $m[u, \cdot]$ *is convex on* V.

The following proposition shows how $x \prec y$ can be verified by using the support function m. Eaton (1984) showed the equivalence between parts (i) and (iii). The equivalence with part (ii) has been established by Giovagnoli and Wynn (1985).

PROPOSITION 2.3. *Let* $x, y \in V$. *The following three statements are equivalent*:

(i) $x \prec y$,
(ii) $f(x) \leqslant f(y)$ *for all* G-*invariant convex functions* $f: V \to \mathbb{R}$,
(iii) $m[u, x] \leqslant m[u, y]$ *for all* $u \in V$.

Various groups of practical interest have the property that the preordering $\prec$ is related to a so-called group-induced cone ordering as defined by Eaton (1987b).

DEFINITION 2.1. The preordering $\prec$ is a group-induced cone ordering if there exists a nonempty closed convex cone $\mathscr{T} \subset V$ such that

(i) $O(x) \cap \mathscr{T}$ is not empty for each $x \in V$,
(ii) $m[u, x] = (u, x)$ for $u, x \in \mathscr{T}$.

Because $x \prec y$ if and only if $g_1 x \prec g_2 y$ for all $g_1, g_2 \in G$, we have on account of part (i) that we only have to characterize $\prec$ on $\mathscr{T}$. For $x, y \in \mathscr{T}$ we obviously have

$$x \prec y \quad \Leftrightarrow \quad (u, x) \leqslant (u, y) \quad \text{for all } u \in \mathscr{T},$$

or, equivalently $(y - x, u) \geqslant 0$ for all $u \in \mathscr{T}$. To put this in other terms, we have $x \prec y$ if and only if $y - x \in \mathscr{T}^*$, the dual cone of $\mathscr{T}$ in V, defined by

$$\mathscr{T}^* = \{v \in V | (v, u) \geqslant 0 \quad \text{for all } u \in \mathscr{T}\}.$$

This fact explains the term "cone ordering", because $\prec$ is a cone ordering on $\mathscr{T}$ defined by the cone $\mathscr{T}^*$:

$$x \prec y \;\Leftrightarrow\; y - x \in \mathscr{T}^* \qquad \text{for} \quad x, y \in \mathscr{T}.$$

This is similar to the ideas in Marshall, Walkup, and Wets (1967).

Groups that are of practical interest give rise to group-induced cone orderings. Moreover, frequently $\mathscr{T}$ is also a polyhedral cone. In this case the elements of $\mathscr{T}$ are generated by taking all possible nonnegative linear combinations of a finite number of elements of some subset T of $\mathscr{T}$. It is said then that T positively generates $\mathscr{T}$. The set T is called a frame for $\mathscr{T}$ if T spans $\mathscr{T}$ positively, but no proper subset of T possesses this property. So, if we have $T = \{t_1, \ldots, t_p\}$, then

$$x \prec y \quad \Leftrightarrow \quad (y - x, t_i) \geqslant 0 \quad \text{for all } i = 1, \ldots, p.$$

We refer to Eaton (1984, 1987b) for concrete examples of cones $\mathscr{T}$ and corresponding frames.

3. FUNDAMENTAL REGIONS, REFLECTION GROUPS

Of special interest to us are the so-called fundamental regions of finite subgroups of $\mathscr{O}(V)$.

DEFINITION 3.1. Let G be a subgroup of $\mathcal{O}(V)$. A subset F of V is called a fundamental region for G in V if and only if

(i) F is open,
(ii) $F \cap gF = \varnothing$ for all $g \neq I$,
(iii) $V = \bigcup_{g \in G} g(\mathrm{cl}\, F)$.

If G is a finite subgroup of $\mathcal{O}(V)$, then there always exists a convex fundamental region F for G in V. This can be established by applying a construction due to Fricke and Klein; see Grove and Benson (1985, Chapter 3). A stronger result has been obtained in the proof of Theorem 2 of Giovagnoli and Wynn (1985). It illustrates the close relationship between a group-induced cone ordering and a fundamental region.

THEOREM 3.1. *Let G be a finite subgroup of $\mathcal{O}(V)$, and F a fundamental region for G in V. Let $x \in F$, and define*

$$\mathscr{T}(x) = \{y \in V | m[x, y] = (x, y)\},$$

$$F(x) = \mathrm{int}\, \mathscr{T}(x).$$

Then $\mathscr{T}(x)$ is a closed convex cone, and $F(x)$ is a fundamental region for G in V.

More specific results regarding fundamental regions can be obtained for reflection groups. The remaining part of this section is devoted to reflection groups; it is completely based on Grove and Benson (1985) and Eaton and Perlman (1977). For some $r \in V$ with $\|r\| = 1$, consider the hyperplane $H_r = \{x \in V\, |(x, r) = 0\}$. The linear transformation S_r will be defined by requiring that $S_r x = x$ for $x \in H_r$ and $S_r x = -x$ for $x \in H_r^{\perp}$. So we have

$$S_r x = x - 2(x, r)r. \tag{3.1}$$

Note that S_r is an orthogonal transformation and $S_r^2 = I$. The transformation S_r is called the reflection through H_r or the reflection along r. The vector r (always with $\|r\| = 1$) is called the root of the reflection S_r. If $S_r \in G$, where G is some subgroup of $\mathcal{O}(V)$, then r is called a root of G. The root system of G is

$$\Delta_G = \{r \in V | S_r \in G\}. \tag{3.2}$$

PROPERTY 3.1. *If $r \in \Delta_G$ and $g \in G$, then $s = gr \in \Delta_G$, and $S_s = gS_r g^{-1}$. If $r \in \Delta_G$, then also $-r \in \Delta_G$.*

DEFINITION 3.2. A reflection group is a closed subgroup G of $\mathcal{O}(V)$ such that there exists a set $\Delta^* \subset V$ with $\|r\| = 1$ for all $r \in \Delta^*$, and G is the smallest closed subgroup of $\mathcal{O}(V)$ containing the reflections $\{S_r \mid r \in \Delta^*\}$.

PROPERTY 3.2. *If G is a reflection group, then G is the closure in $\mathcal{O}(V)$ of the group generated algebraically by $\{S_r \mid r \in \Delta^*\}$. Any reflection group G is generated by $\{S_r \mid r \in \Delta_G\}$.*

An example of an infinite reflection group is the group $\mathcal{O}(V)$. We will only be interested in finite groups. A reflection group G generated by a finite number of reflections is not necessarily finite. A necessary and sufficient condition for G to be finite is that its root system is finite.

PROPERTY 3.3. *Let G be a reflection group generated by a finite set of reflections. If the root system Δ_G is finite, then G is finite.*

Now we consider the fundamental regions of a finite reflection group G. Consider the set

$$T_G = \{t \in V \mid (t, r) \neq 0 \quad \forall r \in \Delta_G\} = \bigcap \{H_r^c \mid r \in \Delta_G\}.$$

So the set T_G is open. Let $t \in T_G$ be arbitrarily chosen. Define the sets of t-positive and t-negative roots:

$$\Delta_t^+ = \{r \in \Delta_G \mid (r, t) > 0\},$$

$$\Delta_t^- = \{r \in \Delta_G \mid (r, t) < 0\}.$$

Obviously we have $\Delta_t^+ = -\Delta_t^-$, $\Delta_G = \Delta_t^+ \cup \Delta_t^-$, and $|\Delta_t^+| = |\Delta_t^-| = \frac{1}{2}|\Delta_G|$; here $|\Delta|$ denotes the number of elements of the set Δ. Now let K_t be the smallest convex cone containing the set Δ_t^+, in symbols

$$K_t = \text{cone}(\Delta_t^+). \tag{3.3}$$

PROPOSITION 3.1. *K_t is a pointed closed convex polyhedral cone in $M_G^\perp$.*

Proof. It is obvious that K_t is a closed convex polyhedral cone. Let $s \in K_t$ with $s \neq 0$. From (3.3) it follows that $(s, t) > 0$, and hence $-s \notin K_t$. So K_t does not contain a linear subspace. ∎

Let $\Pi_t \subset \Delta_t^+$ be a frame for K_t. This means that $K_t = \text{cone}(\Pi_t)$, but no proper subset of Π_t generates K_t. Since K_t is a pointed cone, Π_t equals the set of all extremal rays of K_t. The set Π_t contains exactly m roots $r_1, \dots, r_m$, where $m = \dim M_G^{\perp}$. Moreover, Π_t is a basis for $M_G^{\perp}$, and $(r_i, r_j) \leqslant 0$ for $i \neq j$. The roots $r_1, \dots, r_m$ are called fundamental roots, and the reflections $S_{r_1}, \dots, S_{r_m}$ are called fundamental reflections. The set $\{S_{r_j} \mid j = 1, \dots, m\}$ is a minimal set of reflections generating G. This implies, for instance, that for any $r \in \Delta_G$ there exist $g \in G$ and $i \in \{1, \dots, m\}$ such that $r = gr_i$. The main result is that

$$F_t = \text{int}(\text{dual } K_t)$$

is a fundamental region, and that

$$T_G = \bigcup_{g \in G} gF_t.$$

If $\mathscr{T} = \text{cl } F_t$, then the walls or $(n-1)$-dimensional faces of $\mathscr{T}$ are $\mathscr{T} \cap H_{r_i}$, $i = 1, \dots, m$.

4. G-MAJORIZATION, CONE ORDERINGS, FUNDAMENTAL REGIONS, AND REFLECTION GROUPS

In Section 2 we observed that the cone $\mathscr{T}$ is essential, because if $\mathscr{T}$ is a closed convex polyhedral cone, then the problem of checking $x \prec y$ can be reduced to verifying a finite number of linear inequalities. In a lot of cases the following conditions are satisfied: (1) G is a finite reflection group, (2) the set $\mathscr{T}$ has nonempty interior, (3) $F = \text{int } \mathscr{T}$ is a fundamental region, and (4) $\mathscr{T}$ is a closed convex polyhedral cone with $m[x, y] = (x, y)$ for all $x, y \in \mathscr{T}$. Is this combination of conditions a happy coincidence, or are intrinsic issues involved?

THEOREM 4.1. *Let G be a finite subgroup of $\mathscr{O}(V)$. Then the following statements are equivalent*:

(i) *There exists a closed convex cone $\mathscr{T}$ in V with the property that $m[x, y] = (x, y)$ for all $x, y \in \mathscr{T}$, and $V = \bigcup_{g \in G} g\mathscr{T}$.*

(ii) *There exists a connected fundamental region F with the property that for any $u \in \partial F$ there is a $\tilde{g} \in G$, $\tilde{g} \neq I$, such that $\tilde{g}u = u$.*

(iii) There exists a connected fundamental region F which is unique up to transformations by G within the class of connected fundamental regions.

(iv) G is a reflection group with fundamental region F such that $F = \operatorname{int} \mathscr{T}$, where $\mathscr{T} = \text{dual}(\text{cone}\,\Pi)$, $\Pi = \{r_1, \dots, r_m)$, $m = \dim M_G^{\perp}$, $\|r_i\| = 1$, $S_{r_1}, \dots, S_{r_m}$ are fundamental reflections generating G, and Π is a basis for $M_G^{\perp}$.

(v) There exists a fundamental region F such that $\mathscr{T} = \text{cl}\, F$ is a closed convex cone and for all $x, y \in \mathscr{T}$,

$$x \prec y \quad \Leftrightarrow \quad (x, u) \leqslant (y, u) \quad \forall u \in \mathscr{T}. \tag{4.1}$$

Eaton and Perlman (1977) showed (iv) $\Rightarrow$ (v). Eaton (1984) proved (i) $\Rightarrow$ (v). Giovagnoli and Wynn (1985) established (ii) $\Rightarrow$ (iii) $\Rightarrow$ {(i) and (v)}. The theorem shows the equivalence of the conditions, and it also makes clear that the conditions completely determine the group G. The equivalence of (i), (ii), (iii), and (iv) has been established previously in the author's dissertation (Steerneman 1987). The results (ii) $\Leftrightarrow$ (iv) and (v) $\Rightarrow$ (ii) are believed to be new. The remarks of the referees stimulated the author to try to prove that (v) implies one of the other conditions.

Theorem 3.1 shows that for any finite subgroup of $\mathscr{O}(V)$ there exists a fundamental region F such that $\mathscr{T} = \text{cl}\, F$ is a closed convex cone. Theorem 4.1 shows that G is a reflection group if and only if F is unique up to transformations by the group G. The theorem also illustrates that G-majorization is equivalent to a group-induced cone ordering if and only if G is a reflection group. As a referee remarked, there are examples of infinite groups G where G-majorization is a group-induced cone ordering, but G is not a reflection group. For examples we refer to Eaton (1984, 1987b).

Proof of Theorem 4.1. The following implications will successively be established: (i) $\Rightarrow$ (v) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iv) $\Rightarrow$ (ii). Although the implications (i) $\Rightarrow$ (v), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i) have been proved in the literature, we will also provide their proofs, because they are very short, and for the sake of completeness. Before we start the proof, we give a lemma which may be compared to Lemma 4.3 of Eaton and Perlman (1977).

LEMMA 4.1. *Let A be a subset of V with the property that $m[u, v] = (u, v)$ for all $u, v \in A$. If $x \in A$, $g \neq I$, and $gx \in A$, then $x = gx$.*

Proof of Lemma 4.1. Note that for all $u \in A$ we have

$$(u, gx) \leqslant m[u, x] = (u, x). \tag{4.2}$$

By taking $u = gx$ and $u = x$ respectively in (4.2) we have

$$\|x\|^2 = \|gx\|^2 = (gx, gx) \leqslant (gx, x) \leqslant (x, x) = \|x\|^2.$$

This implies that $(x, gx) = \|x\| \cdot \|gx\|$, and we conclude that $x = gx$, because $\|x\| = \|gx\|$. ■

Now we continue the proof of of the theorem.

(i) $\Rightarrow$ (v): If $x, y \in \mathscr{T}$, then $m[x, u] = (x, u)$ and $m[y, u] = (y, u)$ for all $u \in \mathscr{T}$. It is not difficult to see that we have the following equivalences:

$$\begin{aligned} x \prec y \quad &\Leftrightarrow \quad m[x, u] \leqslant m[y, u] \quad \forall u \in V \\ &\Leftrightarrow \quad m[x, u] \leqslant m[y, u] \quad \forall u \in \mathscr{T} \\ &\Leftrightarrow \quad (x, u) \leqslant (y, u) \quad \forall u \in \mathscr{T}. \end{aligned}$$

(v) $\Rightarrow$ (ii): Let $u \in \partial F$. We have to show that for some $\tilde{g} \in G$, $\tilde{g} \neq I$ we must have $u = \tilde{g}u$. Observe that $u \in \partial(V \backslash \mathrm{cl}\, F)$. There exists a sequence $\{u_n\}$ in the open set $V \backslash \mathrm{cl}\, F$ such that $u_n \to u$. For some $\tilde{g} \neq I$ and an infinite subsequence $\{u_{n_k}\}$ we have $u_{n_k} \in \tilde{g}(\mathrm{cl}\, F)$, because G is finite. Hence $u \in \tilde{g}(\mathrm{cl}\, F)$. So we have $u, \tilde{g}^{-1}u \in \mathrm{cl}\, F$. Since $u \approx \tilde{g}^{-1}u$, we have $(u, v) = (\tilde{g}^{-1}u, v)$ for all $v \in \mathscr{T}$. This implies that $u = \tilde{g}^{-1}u = \tilde{g}u$ with $\tilde{g} \neq I$.

(ii) $\Rightarrow$ (iii): The existence of F being obvious, we have to establish its uniqueness in the sense that any other connected fundamental region F' is of the form gF for some $g \in G$. We shall first establish that $F' \cap \partial gF = \varnothing$ for all $g \in G$. If not, then there would exist $u \in F' \cap \partial gF$. Since $g^{-1}u \in \partial F$, there exists some $\tilde{g} \in G$, $\tilde{g} \neq I$ with $\tilde{g}g^{-1}u = g^{-1}u$. So we have $\tilde{\tilde{g}}u = u$ with $\tilde{\tilde{g}} = g\tilde{g}g^{-1} \neq I$. But $u, \tilde{\tilde{g}}u \in F'$ is impossible for a fundamental region unless $\tilde{\tilde{g}} = I$. Hence $F' \cap \partial gF = \varnothing$ for all $g \in G$. As F' is connected, we must have $F' \subset gF$ for some $g \in G$. Since F' and gF are open, we cannot have $V = \bigcup_{g \in G} g(\mathrm{cl}\, F)$ unless $F' = gF$.

(iii) $\Rightarrow$ (i): Let $x \in F$. Define $\mathscr{T}(x)$ and $F(x)$ as in Theorem 3.1. So $F(x)$ is a fundamental region. From the uniqueness of F it follows that $F(x) = g^*F$ for some $g^* \in G$. Since $x \in F(x)$ and $x \in F$, we have $F(x) = F$ and $\mathscr{T}(x) = \mathrm{cl}\, F$, which holds for any $x \in F$. It now easily follows that $m[x, y] = (x, y)$ for all $x, y \in \mathrm{cl}\, F = \mathscr{T}$, where $\mathscr{T}$ is a closed convex cone.

(iv) $\Rightarrow$ (ii): If $u \in \partial F$, then u is an element of an $(n-1)$-dimensional face of $\mathscr{T}$, i.e. $u \in \mathscr{T} \cap H_r$ for some $r \in \Pi$, where $H_r = \{y \in V \mid (r, y) = 0\}$. Now we have $S_r x = x$ for all $x \in H_r$.

(ii) $\Rightarrow$ (iv): Let K be the dual cone of $\mathscr{T}$, where $\mathscr{T}$ is defined as in (i). Of course K is a closed convex cone. K is also pointed, because otherwise a linear subspace L exists with $\dim L \geqslant 1$, such that $L \subset K$. This would imply that $\mathscr{T} = \operatorname{dual} K \subset L^{\perp}$, which has empty interior. This is impossible. The proof will be completed by establishing the following claims.

Claim 1: K is polyhedral.

Claim 2: If $\Pi = \{r_1, \dots, r_p\}$ with $\|r_i\| = 1$ is a frame of K, then $S_r \in G$ for all $r \in \Pi$.

Claim 3: If H is the finite reflection group generated by $\{S_r \mid r \in \Pi\}$, then $H = G$.

Proof of Claim 1. We know that $C(x) = \operatorname{conv}\{gx \mid g \in G\}$ is a closed convex polytope, because G is finite. We have to relate $C(x)$ to K. The following lemma will be needed.

LEMMA 4.2. *Let $\mathscr{T}$ be defined as in part* (i), *and let $x \in \mathscr{T}$. Then $C(x) = \cap_{g \in G} g(x - K)$.*

Proof. Let $y \in C(x)$; then $\tilde{g}y \in \mathscr{T}$ for some $\tilde{g} \in G$. Since $\tilde{g}y \prec x$ and $x, \tilde{g}y \in \mathscr{T}$, we have by (v) that

$$(y, t) \leqslant m[y, t] = m[\tilde{g}y, t] = (\tilde{g}y, t) \leqslant (x, t)$$

for all $t \in \mathscr{T}$. [Note that we have already showed that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v).] This implies that $(x - y, t) \geqslant 0$ for all $t \in \mathscr{T}$, and hence $x - y \in K$, or, equivalently, $y \in x - K$. So we may conclude $C(x) \subset x - K$. From the G-invariance of $C(x)$ it follows that $C(x) \subset \cap_{g \in G} g(x - K)$. The converse can be established as follows. Let $y \in \cap_{g \in G} g(x - K)$. For some $\tilde{g} \in G$ we have $y \in \tilde{g}\mathscr{T}$. So $x, \tilde{g}^{-1}y \in \mathscr{T}$ and $x - \tilde{g}^{-1}y \in K$. On account of (v) it follows that $\tilde{g}^{-1}y \prec x$, which implies $y \prec x$. ■

This lemma may be compared with Corollary 4.1 of Eaton and Perlman (1977). We apply it in order to show that K is a polyhedral cone. For $x \in F$ arbitrarily chosen, define

$$K_x = \operatorname{cone}(x - C(x)) = \operatorname{cone}\{x - gx \mid g \in G\},$$

which is a closed convex polyhedral cone, because G is finite. We establish that $K_x = K$, which completes the proof of claim 1. From Lemma 4.2 it is obvious that $K_x \subset K$. We focus on the proof of $K \subset K_x$. For $x \in \mathcal{T}$ we have that $C(x) \cap g\mathcal{T} = g(\mathcal{T} \cap (x - K))$ for all $g \in G$. This can be seen as follows:

$$\begin{aligned} y \in C(x) \cap g\mathcal{T} \quad &\Leftrightarrow \quad g^{-1}y \in C(x) \cap \mathcal{T} \\ &\Leftrightarrow \quad x, g^{-1}y \in \mathcal{T} \text{ and } x - g^{-1}y \in K \\ &\Leftrightarrow \quad y \in g(\mathcal{T} \cap (x - K)). \end{aligned}$$

By Lemma 4.2 we now have

$$C(x) = \bigcup_{g \in G} g(\mathcal{T} \cap (x - K)).$$

Here $x \in (\operatorname{int} \mathcal{T}) \cap (x - K)$, and $x \notin g(\mathcal{T} \cap (x - K))$ for any $g \neq I$. If not, let $y \in K$; then for $\lambda \in (0,\infty)$ small enough we have $x - \lambda y \in (\operatorname{int} \mathcal{T}) \cap (x - K)$. Hence $x - \lambda y \in C(x)$. This implies that $\lambda y \in x - C(x) \subset K_x$ and therefore $y \in K_x$. So we have $K = K_x$, which is convex polyhedral cone. ■

Proof of Claim 2. Since K is a pointed closed convex polyhedral cone, K has a frame $\Pi = \{r_1, \dots, r_p\}$ with $\|r_i\| = 1$, such that $\{\lambda r_i \mid \lambda \geqslant 0\}$ are the extremal rays or the 1-dimensional faces. There is one-one correspondence between the extremal rays of K and the $(n-1)$-dimensional faces or walls of $\mathcal{T}$. The $(n-1)$-dimensional faces of $\mathcal{T}$ are

$$\mathcal{T}_i = \operatorname{cl}\{u \in V | (u, r_i) = 0,\ (u, r_j) > 0 \text{ for } j \neq i\}.$$

We have to show that $S_{r_i} \in G$ for $i = 1, \dots, p$. Let u be any point in the relative interior of $\mathcal{T}_i$. There exists a $\tilde{g} \in G$ with $\tilde{g} \neq I$ such that $\tilde{g}u = u$. We have $u \in \mathcal{T} \cap \tilde{g}\mathcal{T}$, but $\operatorname{int} \mathcal{T} \cap \operatorname{int} \tilde{g}\mathcal{T} = \varnothing$. So there exists a separating hyperplane $H_r = \{x | (x, r) = 0\}$, such that

$$\mathcal{T} \subset \overline{H}_r^+ = \{x \in V | (x, r) \geqslant 0\},$$

$$\tilde{g}\mathcal{T} \subset \overline{H}_r^- = \{x \in V | (x, r) \leqslant 0\}.$$

This implies that

$$(u, r) = (\tilde{g}u, r) \leqslant 0 \leqslant (u, r).$$

So $u \in \mathscr{T} \cap \tilde{g}\mathscr{T} \cap H_r$. Obviously H_r is a supporting hyperplane for $\mathscr{T}$, but also for $\tilde{g}\mathscr{T}$. Recall that $u \in \operatorname{ri} \mathscr{T}_i$ (the relative interior of $\mathscr{T}_i$ with respect to H_{r_i}), where $\mathscr{T}_i$ is an $(n-1)$-dimensional face. This implies that a supporting hyperplane through u for $\mathscr{T}$ is uniquely determined and hence $H_r = H_{r_i}$. Clearly we have $r = \pm r_i$. Similarly $\tilde{g}\mathscr{T}_i$ is an $(n-1)$-dimensional face of $\tilde{g}\mathscr{T}$, and therefore $H_r = H_{\tilde{g}r}$ is the unique supporting hyperplane through $u = \tilde{g}u$ for $\tilde{g}\mathscr{T}$. So, we must have

$$\tilde{g}\mathscr{T} \subset \bar{H}^+_{\tilde{g}r} = \bar{H}^-_r .$$

This implies $\tilde{g}r = -r$. Recall that $r = \pm r_i$. We conclude that $\tilde{g}r_i = -r_i$. It remains to show that $\tilde{g} = S_{r_i}$. It suffices to establish that $\tilde{g}x = x$ for $x \in U$, where U is a subset of H_r with nonempty interior in H_r. Now we remark that $\operatorname{ri} \mathscr{T}_i \cap \operatorname{ri} \tilde{g}\mathscr{T}_i = \operatorname{ri}(\mathscr{T}_i \cap \tilde{g}\mathscr{T}_i)$, where the relative interior is taken with respect to H_r. Hence $U = \mathscr{T}_i \cap \tilde{g}\mathscr{T}_i$ has nonempty interior in H_r. For any $x \in U$ we have $x \in \mathscr{T}$ and $x \in \tilde{g}\mathscr{T}$. This means that $x, \tilde{g}^{-1}x \in \mathscr{T}$. On account of Lemma 4.1 and (i) (which can now be applied) we have $x = \tilde{g}^{-1}x = \tilde{g}x$. Therefore $\tilde{g} = S_{r_i}$. ∎

Proof of Claim 3. Let H be the group generated by the reflections $\{S_{r_1}, \dots, S_{r_p}\}$, and let Δ_H be the set of all roots of H. Obviously Δ_H is a finite set. Define $T_H = \{t \in V \mid (r,t) \neq 0 \ \forall r \in \Delta_H\}$, and let $t \in F \cap T_H$ be arbitrarily chosen. Since $t \in F = \operatorname{int}(\operatorname{dual} K)$, we have $(t, r) > 0$ for all $r \in \Pi$.

Define $\Delta_t^+ = \{r \in \Delta_H \mid (t, r) > 0\}$. It is clear that $\Pi \subset \Delta_t^+$. Now we define $K_t = \operatorname{cone}(\Delta_t^+)$, and we have $K \subset K_t$. This implies that $\mathscr{T}_t = \operatorname{dual} K_t \subset \operatorname{dual} K = \mathscr{T}$. Define $F_t = \operatorname{int} \mathscr{T}_t$. According to the theory of fundamental regions for finite reflection groups (see Section 2), F_t is a fundamental region for H. Since $H \subset G$ and $\mathscr{T}_t \subset \mathscr{T}$, we have

$$V = \bigcup_{g \in H} g\mathscr{T}_t \subset \bigcup_{g \in G} g\mathscr{T} = V. \qquad ∎$$

Hence we must have $\mathscr{T}_t = \mathscr{T}$ and $H = G$.

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